

CARROUSEL IN FAMILY AND NON-ISOLATED HYPERSURFACE SINGULARITIES IN \mathbb{C}^3

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ABSTRACT. We study the boundary L_t of the Milnor fiber for the reduced holomorphic germs $f : (\mathbb{C}^3, 0) \rightarrow (\mathbb{C}, 0)$ having a non-isolated singularity at 0. We prove that L_t is a graph manifold by using a new technique of carrousel depending on one parameter varying on a circle. Our results enable one to compare the topology of L_t and of the link of the normalization of $f^{-1}(0)$.

1. INTRODUCTION

We denote by \mathbb{B}_r^{2n} the $2n$ -ball with radius $r > 0$ centered at the origin of \mathbb{C}^n and by \mathbb{S}_r^{2n-1} the boundary of \mathbb{B}_r^{2n} .

Let $f : (\mathbb{C}^3, 0) \rightarrow (\mathbb{C}, 0)$ be a reduced holomorphic germ. The singularity of f at 0 is allowed to be non-isolated. We consider the three underlying topological objects :

- The link $L_0 = f^{-1}(0) \cap \mathbb{S}_\epsilon^5$ of the surface $f^{-1}(0)$ at 0, whose homeomorphism class of L_0 does not depend on ϵ when $\epsilon > 0$ is sufficiently small ([16], [1]).
- The boundary $L_t = f^{-1}(t) \cap \mathbb{S}_\epsilon^5$ of the Milnor fiber of f , where $0 < |t| < \epsilon$, whose diffeomorphism class does not depend on t when $|t|$ is sufficiently small ([16], [6]).
- The link \overline{L}_0 of the normalization of the surface $F_0 = f^{-1}(0) \cap \mathbb{B}_\epsilon^6$ at 0, which can be defined up to diffeomorphism by $\overline{L}_0 = n^{-1}(L_0)$, where $n : \overline{F}_0 \rightarrow F_0$ denotes the normalization morphism of F_0 ([3]).

When the origin is an isolated singular point, L_0 , L_t and \overline{L}_0 are three-dimensional differentiable manifolds, each of them being diffeomorphic to the others.

In this paper, we assume that the singular locus $\Sigma(f)$ of f is 1-dimensional. Then only L_t and \overline{L}_0 are differentiable manifolds.

The resolution theory implies that \overline{L}_0 is a graph manifold in the sense of Waldhausen, or equivalently a plumbed manifold in the sense of Neumann ([17], [19], [22]). More precisely, the plumbing graph of

\overline{L}_0 is given, in its normal form, as the dual graph of a good minimal resolution of the normal surface singularity \overline{F}_0 .

We will not recall here the notions of Seifert, graph and plumbed manifolds. For a quick survey adapted to our situation, see e.g. [15], section 3.

In [14], we study irreducible (i.e. $\gcd(m, k, l) = 1$) non-isolated (i.e. $1 \leq k \leq l$ and $2 \leq l$) Hirzebruch hypersurface singularities in \mathbf{C}^3 given by the equation $z^m - x^k y^l = 0$. We show that the boundary L_t of the Milnor fiber is always a Seifert manifold and we give an explicit description of the Seifert structure.

In [15] we study the L_t for the non-isolated singularities in \mathbf{C}^3 with equation $z^m - g(x, y) = 0$ where $m \geq 2$ and $g(x, y) = 0$ is a non-reduced plane curve germ. For these germs, defined by $z^m - g(x, y) = 0$, we show that the connected components of the vanishing zone M_t (defined, in this paper, in 2.10) has a Seifertic structure and we describe it in details.

More recently, in [18], A. Némethi and A. Szilárd describe the boundary of the Milnor fiber for other families of examples.

In [13], we state with a sketch of proof that for a germ $f : (\mathbf{C}^3, 0) \longrightarrow (\mathbf{C}, 0)$, the boundary L_t of the Milnor fiber is also a graph manifold. The main aim of this paper is to give a detailed proof of this result.

We first describe the manifold L_t using the following strategy (Section 2) : by hypothesis the singular locus of f , $\Sigma(f)$, is a curve. Let $K_0 = L_0 \cap \Sigma(f)$ be the link of the singular locus in L_0 . Let $\overline{K}_0 = n^{-1}(K_0)$ be the pull-back of K_0 in \overline{L}_0 and $\overline{\Sigma}(f) = n^{-1}(\Sigma(f))$ be the pull-back of the singular locus. A good resolution of the pair $(\overline{F}_0, \overline{\Sigma}(f))$ provides a Waldhausen decomposition for \overline{L}_0 as a union of Seifert manifolds such that \overline{K}_0 is a union of Seifert leaves. Let \overline{M}_0 be a tubular neighborhood of \overline{K}_0 in \overline{L}_0 . The closure \overline{N}_0 of $(\overline{L}_0 \setminus \overline{M}_0)$ is an irreducible Waldhausen graph manifold with boundary that we called the trunk of \overline{L}_0 .

On the other hand, we define a submanifold M_t of L_t called the vanishing zone around K_0 . And we obtain the following theorem:

Theorem.

- (1) The closure N_t of $L_t \setminus M_t$ is orientation preserving diffeomorphic to the trunk \overline{N}_0 .
- (2) The manifold M_t is an irreducible Waldhausen graph manifold.

Remark: The irreducibility of M_t is justify in lemma 5.3. Moreover, our description of L_t enables us to compare (in the proof of theorem

5.1) the normalized plumbing graph of L_t with the minimal resolution graph of the pair $(\overline{F}_0, \overline{\Sigma}(f))$.

Sections 3 and 4 are devoted to the proof of point (2). The key idea consists of describing a Waldhausen decomposition of M_t in terms of a “carrousel in family” (see (4.7) to (4.17)). Let us recall that the “carrousel” has been introduced by D.T.Lê, in [10] p.163, to obtain a geometric proof of the monodromy theorem.

A key tool in the implementation of this carrousel process is a parametrization result presented in Section 3 (Theorem 3.2) which enables one to obtain a suitable parametrization of the critical locus of a projection from M_t to a solid tori (Lemma 4.6).

The main aim in the study of the topological aspects of singularities consists of describing the analytical properties of a singularity which can be characterized through some topological underlying objects. One of the most important results in this direction is the following famous theorem of Mumford, which gives a topological characterization of a smooth point on a normal surface :

Theorem 1.1. ([17]) *Let $(X, 0)$ be a germ of normal complex surface. If the link L_0 of $(X, 0)$ has the homotopy type of the 3-sphere, then 0 is a smooth point of X .*

The initial motivation of this work was to prove an analogous theorem for non-isolated singularities of hypersurfaces in \mathbb{C}^3 . One of the advantages of our description of L_t as the gluing of the trunk and the vanishing zone is that it makes fairly easy the comparison of L_t with \overline{L}_0 in most of the cases. We obtain the following topological characterization of isolated singularities for the analytic reduced germs $f : (\mathbb{C}^3, 0) \longrightarrow (\mathbb{C}, 0)$ (5.1).

Theorem. Let $f : (\mathbb{C}^3, 0) \longrightarrow (\mathbb{C}, 0)$ be a reduced holomorphic germ whose singular locus is at most 1-dimensional. We assume that either f is reducible or L_t is not a lens space. Then the following assertions are equivalent.

- (i) f is either smooth or has an isolated singularity at 0.
- (ii) The boundary $\overline{L_t}$, $t \neq 0$ of the Milnor fibre of f is homeomorphic to the link \overline{L}_0 of the normalization of $f^{-1}(0)$.

In [15], for the germs with equations $z^m - g(x, y) = 0$ where $m \geq 2$ and $g(x, y) = 0$ is a non-reduced plane curve germ, we proved that L_t is never homeomorphic to \overline{L}_0 even if L_t is a lens space, and that the later case arises if and only if $m = 2$ and g has the analytic type of xy^l .

The problem of the characterization of the germs $f : (\mathbb{C}^3, 0) \longrightarrow (\mathbb{C}, 0)$, which do not have the analytic type of germs with equation $z^m - g(x, y) = 0$, and which have a lens space as boundary L_t of their Milnor fiber, remains open. Proposition 5.2 shows that this open case concerns a very special family of singularities.

2. THE TRUNK AND THE VANISHING ZONE.

In this section, we define the trunk and the vanishing zone of L_t . As a preliminary, we start in (2.1) and (2.2) by performing generic choices of the coordinates axis in \mathbb{C}^3

2.1. The Weierstrass preparation theorem implies that we can choose f in $\mathbb{C}\{x, y\}[z]$. Then, the intersection Γ_0 between $\{f = 0\}$ and the hypersurface in \mathbb{C}^3 with equation $\{\frac{\partial f}{\partial z} = 0\}$ is a curve which contains $\Sigma(f)$.

Claim For a generic choice of the x -axis, $(\{\frac{\partial f}{\partial z} = 0\} \cap \{\frac{\partial f}{\partial y} = 0\})$ does not meet the boundary of the Milnor fiber and:

$$\Sigma(f) = \Gamma_0 \cap \{\frac{\partial f}{\partial y} = 0\}.$$

Proof. D.T. Lê and B.Teissier (for example see (2.2.2) in [9] or IV.1.3.2 p.420 in [21]) have proved that, for a generic choice of the x -axis,

$$(\{\frac{\partial f}{\partial z} = 0\} \cap \{\frac{\partial f}{\partial y} = 0\}) = (\Sigma(f) \cup \Gamma_{(x,f)}),$$

where the irreducible components of $\Gamma_{(x,f)}$ are one-dimensional and not included in $\{f = 0\}$. They have called $\Gamma_{(x,f)}$ the *polar curve of f for the direction x* . Then, the boundary of the Milnor fiber does not meet $\Gamma_{(x,f)}$ (but its interior does). Moreover, the Milnor fiber does not meet $\Sigma(f)$.

□

2.2. Let $P : \mathbb{C}^3 \longrightarrow \mathbb{C}^2$ be the map defined by

$$P(x, y, z) = (x, y).$$

Let Δ_0 be $P(\Gamma_0)$, Δ_0 is the *discriminant curve*. Perhaps after performing a linear change of coordinates in \mathbb{C}^2 , we can assume that the x -axis is, at the origin, transverse to Δ_0 and that in \mathbb{C}^3 , the hyperplanes $X_a = \{x = a\}$ meet Γ_0 transversely around the origin.

2.3. For technical reasons, we replace in this paper the standard Milnor ball \mathbb{B}_ϵ^6 by a polydisc

$$B(\alpha) = \mathbb{B}_\alpha^2 \times \mathbb{B}_\beta^2 \times \mathbb{B}_\gamma^2 = \{(x, y, z) \in \mathbb{B}_\epsilon^6, \quad |x| \leq \alpha, |y| \leq \beta, |z| \leq \gamma\}$$

where $0 < \alpha < \beta < \gamma < \epsilon/3$.

Definition. The polydisc $B(\alpha)$ is a *Milnor polydisc for f* if for each α' with $0 < \alpha' \leq \alpha$,

- (1) the pair $(B(\alpha'), f^{-1}(0) \cap B(\alpha'))$ is diffeomorphic to the pair $(\mathbb{B}_\epsilon^6, f^{-1}(0) \cap \mathbb{B}_\epsilon^6)$,
- (2) there exists η with $0 < \eta < \alpha'$ such that:
 - (a) the restriction of f to $W(\alpha', \eta) = B(\alpha') \cap f^{-1}(\mathbb{B}_\eta^2 \setminus \{0\})$ is a locally trivial differentiable fibration over $\mathbb{B}_\eta^2 \setminus \{0\}$,
 - (b) the isomorphism class of this fibration does not depend on α' and η when $0 < \eta < \alpha' \leq \alpha$.

Let us denote by S the boundary of $B(\alpha)$ and let $S(\alpha)$ be the subset of S defined by $S(\alpha) = \mathbb{S}_\alpha^1 \times \text{int}(\mathbb{B}_\beta^2) \times \text{int}(\mathbb{B}_\gamma^2)$. We can choose $0 < \alpha < \beta < \gamma < \epsilon/3$ such that the two following inclusions hold :

- (1₀) $(S \cap f^{-1}(0)) \subset \{|z| < \gamma\}$, and
- (2₀) $(\Gamma_0 \cap S) \subset S(\alpha)$.

According to [9], Section 1, the generic choice of coordinates axis performed in (2.1) and (2.2) and the above conditions on α, β, γ imply that the polydisc $B(\alpha)$ is a Milnor polydisc for f .

In the sequel, we will then replace the objects defined in the introduction by the following :

- For $0 \leq |t| \leq \eta$,

$$F_t = f^{-1}(t) \cap B(\alpha) \text{ and } L_t = F_t \cap S,$$

- $\overline{L_0} = n^{-1}(L_0)$, where $n : \overline{F_0} \rightarrow F_0$ denotes the normalization of F_0 ,
- $K_0 = \Sigma(f) \cap L_0$ and $\overline{K_0} = n^{-1}(K_0)$.

Remark 2.4. Let us denote by S' the boundary of $\mathbb{B}_\alpha^2 \times \mathbb{B}_\beta^2$. The restriction $P_0 : L_0 \rightarrow S'$ of P on $L_0 = S \cap f^{-1}(0)$ is a ramified cover whose ramification locus is the algebraic link $\Delta_0 \cap S'$ and whose generic order is the degree of f in z .

The above construction implies the following proposition.

Proposition 2.5. *For a sufficiently small tubular neighborhood V of $\Delta_0 \cap S'$, the two following conditions hold :*

- (1) $V \subset \mathbb{S}_\alpha^1 \times \text{int}(\mathbb{B}_\beta^2)$.
- (2) *Let M_0 be the union of the connected components of $P_0^{-1}(V)$ which contain the components of the link K_0 . Then $\overline{M_0} = n^{-1}(M_0)$ is a tubular neighborhood of $\overline{K_0}$ in $\overline{L_0}$.*

Definition 2.6. The *trunk* of L_0 is the closure N_0 of $L_0 \setminus M_0$ in L_0 . The *trunk* of $\overline{L_0}$ is the closure $\overline{N_0}$ of $\overline{L_0} \setminus \overline{M_0}$ in $\overline{L_0}$.

Proposition 2.7. *The trunk N_0 is a Waldhausen graph manifold with boundary.*

Proof. By definition $\overline{N_0} = n^{-1}(N_0)$. By construction N_0 does not meet the singular locus $\Sigma(f)$. Therefore the restriction of n on $\overline{N_0}$ is a diffeomorphism from $\overline{N_0}$ to N_0 . A good resolution of the pair $(\overline{F_0}, n^{-1}(\Sigma(f)))$ provides a Waldhausen decomposition for $\overline{L_0}$ as a union of Seifert manifolds such that $\overline{K_0}$ is a union of Seifert leaves. As $\overline{M_0}$ is a tubular neighborhood of $\overline{K_0}$ in $\overline{L_0}$, then the closure $\overline{N_0}$ of $(\overline{L_0} \setminus \overline{M_0})$ is a waldhausen graph manifold with boundary. \square

Corollary 2.8. *The number of boundary components of the trunk N_0 is equal to the number of irreducible components of the curve $\overline{\Sigma(f)}$.*

Proof. In the proof of the above proposition, we show that N_0 and $\overline{N_0}$ are diffeomorphic. By construction, the number of boundary components of the trunk $\overline{N_0}$ is equals to the number of connected components of $\overline{K_0}$, which is equal to the number of irreducible components of the curve $\overline{\Sigma(f)}$.

2.9. For each $t \in \mathbb{B}_\eta^2$, the singular set Γ_t of the restriction of P on F_t is the curve

$$\Gamma_t = \left\{ \frac{\partial f}{\partial z} = 0 \right\} \cap F_t,$$

and its discriminant locus is the curve $\Delta_t = P(\Gamma_t)$.

By continuity, we can choose η sufficiently small, $0 < \eta \ll \alpha$, in such a way that for each t , $|t| \leq \eta$, the properties that we already have for $t = 0$, hold for $t \in \mathbb{B}_\eta^2$, i.e. :

- (1_t) $L_t \subset \{|z| < \gamma\}$
- (2_t) Γ_t is a curve which intersects transversally S inside $S(\alpha)$

Moreover, let $P_t : L_t \rightarrow S'$ the restriction of P to L_t . Then,

- (3_t) the map $P_t : L_t \rightarrow S'$ is a finite ramified cover with ramification locus $\Gamma_t \cap S(\alpha)$ and branching locus $\Delta_t \cap S'$.
- (4_t) $\Delta_t \cap S' \subset \text{int}(V)$.

Definition 2.10. Let $L(\eta) = f^{-1}(\mathbb{B}_\eta^2) \cap S$ and let $M(\eta)$ be the union of the connected components of $L(\eta) \cap P^{-1}(V)$ which contain K_0 . For any $t \in \mathbb{B}_\eta^2$, let $M_t = M(\eta) \cap L_t$. By definition M_t is the *vanishing zone* of L_t and the closure N_t of $L_t \setminus M_t$ in L_t is the *trunk* of L_t .

Notice that the choice of V (see 2.5), implies that $M(\eta) \subset S(\alpha)$.

Proposition 2.11. *Let $N(\eta)$ be the closure of $L(\eta) \setminus M(\eta)$ in $L(\eta)$. There exists a sufficiently small η such that f restricted to $N(\eta)$ is a fibration on \mathbb{B}_η^2 .*

Corollary 2.12. *There exists a sufficiently small η such that for all $t \in \mathbb{B}_\eta^2 \setminus \{0\}$, N_t is orientation preserving diffeomorphic to N_0 .*

Proof of Proposition 2.11.

i) Let

$$\Gamma(\eta) = L(\eta) \cap \left\{ \frac{\partial f}{\partial z} = 0 \right\}.$$

Then, the restriction of (P, f) on $S \setminus \Gamma(\eta)$ is a submersion. By (4_t), in (2.9), $\Gamma(\eta)$ does not meet the boundary of $N(\eta)$, hence the restriction of f on the boundary of $N(\eta)$ is a fibration.

ii) Let γ' such that $0 < \gamma' < \gamma$. In S , we consider $\bar{S}(\alpha) = \mathbb{S}_\alpha^1 \times \mathbb{B}_\beta^2 \times \mathbb{B}_{\gamma'}^2$, and $\bar{S}(\beta) = \mathbb{B}_\alpha^2 \times \mathbb{S}_\beta^1 \times \mathbb{B}_{\gamma'}^2$, where $\alpha < \beta < \gamma' < \gamma$.

As $L(\eta)$ is compact, (1_t) implies that there exists γ' and η with $0 < \eta \ll \alpha < \beta < \gamma' < \gamma$ such that for all t with $0 \leq |t| \leq \eta$,

$$L_t \subset (\bar{S}(\alpha) \cup \bar{S}(\beta)).$$

By (2_t) in (2.9), $\Gamma(\eta)$ does not meet $\bar{S}(\beta)$, hence the restriction of f on $N(\eta) \cap \bar{S}(\beta)$ is a fibration.

iii) Now, we have to prove that the restriction of f on $N(\eta) \cap \bar{S}(\alpha)$ is a fibration. Points i) and ii) show that it is a fibration on its boundary. So, it is sufficient to prove that the projection on the x axis is transverse to f on $N(\eta) \cap S(\alpha)$ i.e. to prove that there exists a sufficiently small $\eta > 0$ such that the set

$$A = N(\eta) \cap S(\alpha) \cap \left\{ \frac{\partial f}{\partial z} = 0 \right\} \cap \left\{ \frac{\partial f}{\partial y} = 0 \right\}$$

is empty. But for a general choice of the coordinates x and y , lemma (2.1) implies that:

$$L_0 \cap \left\{ \frac{\partial f}{\partial z} = 0 \right\} \cap \left\{ \frac{\partial f}{\partial y} = 0 \right\} = K_0 \subset \text{int}(M_0)$$

Then, by continuity :

$$(*) \quad L(\eta) \cap \left\{ \frac{\partial f}{\partial z} = 0 \right\} \cap \left\{ \frac{\partial f}{\partial y} = 0 \right\} \subset \text{int}(M(\eta))$$

(*) implies that A is empty. \square

2.13. Now, let us describe more precisely the connected components of the vanishing zone M_t .

The tubular neighborhood V of $\Delta_0 \cap S'$, used above to obtain the vanishing zone, can be defined as follows:

Let $\delta_1, \dots, \delta_s$ be the irreducible components of Δ_0 . Let us fix $i \in \{1, \dots, s\}$, and let

$$u \mapsto (u^k, \phi_i(u)), \text{ where } \phi_i(u) = \sum_{j=1}^{\infty} a_j u^j$$

be a Puiseux expansion of the branch δ_i of Δ_0 . Let us consider the neighborhood W_i of δ_i in \mathbb{C}^2 defined by

$$W_i = \{(x, y) \in \mathbb{C}^2 \mid x = u^k, |y - \phi_i(u)| \leq \theta, u \in \mathbb{C}\},$$

where θ is a positive real number.

We now choose θ sufficiently small, $0 < \theta \ll \alpha$, in such a way that:

- (1) for each $i = 1, \dots, s$, W_i intersects transversally S' inside $\mathbb{S}_\alpha^1 \times \text{int}(\mathbb{B}_\beta^2)$,
- (2) the intersection $V_i = W_i \cap S'$ is a tubular neighborhood of the knot $\delta_i \cap S'$,
- (3) the solid tori V_i are disjoint.

Let $V = \bigcup_{i=1}^s V_i$. By continuity there exists $\eta \ll \theta$ such that for each t , $|t| \leq \eta$, one has $(\Delta_t \cap S') \subset \text{int}(V)$.

Let σ be an irreducible component of $\Sigma(f)$. There exists $i \in \{1, \dots, s\}$ such that $P(\sigma) = \delta_i$. We denote by $M(\eta, \sigma)$ the connected component of $P^{-1}(V_i) \cap L(\eta)$ which contains the knot $K_0(\sigma) = \sigma \cap S$ of σ in S . Define

$$M_t(\sigma) = M(\eta, \sigma) \cap L_t.$$

The three-dimensional manifold $M_t(\sigma)$ is connected, and we obtain :

$$M_t = \bigcup_{j=1}^r M_t(\sigma_j),$$

where $\{\sigma_j, 1 \leq j \leq r\}$ is the set of the irreducible components of $\Sigma(f)$.

For each $j = 1, \dots, r$, let \bar{r}_j be the number of irreducible components of the curve $n^{-1}(\sigma_j)$. The boundary of $M_t(\sigma_j)$ consists of \bar{r}_j tori.

Definition. $M_t(\sigma)$ is the *vanishing zone* of L_t along σ .

Proposition (2.7), Corollary (2.12), and the construction 2.13 summarize in the following theorem :

Theorem 2.14. (1) *The boundary L_t of the Milnor fiber of f decomposes as the union*

$$L_t = N_t \cup M_t,$$

- (2) *$N_t \cap M_t$ is a disjoint union of r tori, where r is the number of irreducible components of the curve $\overline{\Sigma(f)}$,*
- (3) *N_t is a Waldhausen manifold orientation preserving diffeomorphic to the trunk N_0 ,*
- (4) *Let $\sigma_1, \dots, \sigma_r$ be the union of irreducible components of $\Sigma(f)$. The connected components of the vanishing zone M_t are the manifolds $M_t(\sigma_j), j = 1 \dots r$.*

Corollary 2.15. *The manifold L_t is connected.*

Proof. The number of connected components of \overline{F}_0 and \overline{L}_0 is equal to the number of irreducible components of f . The intersection between two irreducible components of $f = 0$ furnishes at least one irreducible component of the singular locus $\Sigma(f)$ and a corresponding connected component of the vanishing zone. Hence, the constructions given here show that after the gluing of all connected components of the vanishing zone with the trunk, we obtain a connected manifold L_t .

□

Remark 2.16. Corollary 2.15 implies that the Milnor fiber F_t is connected. As the singular locus of f has dimension 1, F_t is connected by a much more general result of M. Kato and Y. Matsumoto in [8].

Remark 2.17. To prove that L_t is a Waldhausen graph manifold, we still have to prove that $M_t(\sigma)$ is a waldhausen graph manifold for any irreducible component σ of $\Sigma(f)$. This will be done in Section 4.

3. A PARAMETRIZATION THEOREM

In this section, we consider an analytic germ $h : (\mathbb{C}^3, 0) \rightarrow (\mathbb{C}, 0)$ such that $\forall x \in \mathbb{C}, h(x, 0, 0) = 0$, and the germ of hypersurface H with equation $h = 0$. We suppose that h is reduced and irreducible.

For each $x \in \mathbb{C}$, we denote by $h_x : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}, 0)$ the germ defined by $h_x(y, z) = h(x, y, z)$. Hence h_x has an isolated singular point at $(x, 0, 0)$ for all $x \in \mathbb{B}_\alpha^2 \setminus \{0\}$.

Let us fix $\alpha < 1$ and $\epsilon < \alpha$ such that for each $x \in \mathbb{S}_\alpha^1$, $\{x\} \times \mathbb{B}_\epsilon^4$ is a Milnor ball for the germ of curve $h_x = 0$ at $(x, 0, 0)$.

Definition 3.1. A branch of H along \mathbb{S}_α^1 is the adherence of a connected component of the intersection $H \cap (\mathbb{S}_\alpha^1 \times (\mathbb{B}_\epsilon^4 \setminus \{0\}))$.

Theorem 3.2. Let G be a branch of H along \mathbb{S}_α^1 . There exists d, i and $j \in \mathbb{N}^*$, and two convergent power series $b(x^{1/d}, u) \in \mathbb{C}\{x^{1/d}\}\{u\}$ and $c(x^{1/d}, u) \in \mathbb{C}\{x^{1/d}\}\{u\}$ with $b(x, 0) \neq 0$ and $c(x, 0) \neq 0$, such that

$$(s, u) \longmapsto (s^d, u^i b(s, u), u^j c(s, u))$$

is a parametrization of G .

For each $x \in \mathbb{B}_\alpha^2 \setminus \{0\}$, let $\pi_x : Y_x \rightarrow \{x\} \times \mathbb{B}_\epsilon^4$ be the minimal good resolution of h_x , i.e. the minimal composition of blow-ups of points such that the curve $(h_x \circ \pi_x)^{-1}(0)$ is a normal crossing divisor. We denote by $E_x = \pi_x^{-1}(x, 0, 0)$ the exceptional divisor of π_x .

The proof of Theorem 3.2 will use the following :

Lemma 3.3. Let $h_{1,x}$ be an irreducible component of h_x , let $\tilde{h}_{1,x}$ be its strict transform by π_x and let $P = E_x \cap \tilde{h}_{1,x}$. We can choose local coordinates (u, v) at P in Y_x such that :

- (1) $u = 0$ is a local equation for E_x in Y_x .
- (2) There exist three integers d, i, j in \mathbb{N}^* , two polynomials $\phi(x^{1/d}, u, v)$ and $\psi(x^{1/d}, u, v)$ in $\mathbb{C}\{x^{1/d}\}[u, v]$, where $\phi(x^{1/d}, 0, v)$ and $\psi(x^{1/d}, 0, v)$ are not identically 0, and $s \in \mathbb{B}_{\alpha^{1/d}}^2 \setminus \{0\}$ with $s^d = x$ such that :

$$\pi_x(u, v) = (s^d, u^i \phi(s, u, v), u^j \psi(s, u, v)).$$

- (3) There exist an integer $M \in \mathbb{N}^*$ and two convergent power series $c(x^{1/d}) \in (\mathbb{C}\{x^{1/d}\} \setminus \{0\})$ and $g(x^{1/d}, u, v) \in \mathbb{C}\{x^{1/d}\}\{u, v\}$ such that, for the value s defined just above, we have:

$$(h \circ \pi_x)(u, v) = u^M (ug(s, u, v) + c(s)v)$$

Proof. Let us write $h(x, y, z)$ as the sum

$$h(x, y, z) = \sum_{n=0}^{\infty} c_n(x, y, z),$$

where for all $n \in \mathbb{N}$,

$$c_n(x, y, z) = \sum_{k=0}^n c_{n,k}(x) y^k z^{n-k}$$

with $c_{n,k}(x) \in \mathbb{C}\{x\}$.

Let m be the least integer such that $c_m(x, y, z) \neq 0$. Perhaps after performing a change of variables, one can assume that $c_{m,0}(x) \neq 0$. We start with the blow-up $\pi_{1,x}$ of $(x, 0, 0)$ in \mathbb{C}^3 , i.e.:

$$\pi_{1,x} : Y_{1,x} \rightarrow \{x\} \times \mathbb{B}_\epsilon^4.$$

Let $E_{1,x} = (\pi_{1,x})^{-1}(x, 0, 0)$ be the exceptional divisor of $\pi_{1,x}$. As $c_{m,0}(x) \neq 0$, the axis $y = 0$ is not a line of the tangent cone of h_x . We will write the intersection points $\tilde{h}_x \cap E_{1,x}$ with the help of coordinates (u_1, v_1) given by the standard chart on $(\pi_{1,x})^{-1}(\{x\} \times \mathbb{B}_\epsilon^4)$ defined by

$$\pi_{1,x}(u_1, v_1) = (x, u_1, u_1 v_1)$$

In the local coordinates (u_1, v_1) , we have :

$$(h_x \circ \pi_{1,x})(u_1, v_1) = u_1^m \left(\sum_{k=0}^m c_{m,k}(x) v_1^{m-k} + u_1 g_1(x, u_1, v_1) \right) \quad (*)$$

where

$$g_1(x, u_1, v_1) = \sum_{m'=m+1}^{\infty} u_1^{m'-m-1} c_{m'}(x, u_1, v_1)$$

Then the intersection $\tilde{h}_x \cap E_{1,x}$ consists of the points $(x, 0, v_1)$ such that v_1 is a root of the polynomial

$$Q(v_1) = \sum_{k=0}^m c_{m,k}(x) v_1^{m-k} \in \mathbb{C}\{x\}[v].$$

There exists an integer $e > 0$ such that the decomposition field of the polynomial Q is the fraction field K_e of $\mathbb{C}\{x^{1/e}\}$ (for example see D.Eisenbud [4], p.295). There exists a unique root $r_1 \in K_{d_1}$ of Q , where $d_1 \leq e$ is the minimal integer such that $r_1 \in K_{d_1}$, and a complex number s_1 which satisfies $s_1^{d_1} = x$, such that the strict transform of $h_{1,x}$ (by $\pi_{1,x}$), cuts $E_{1,x}$ at the point $P_1 = (0, r_1(s_1))$. The strict transform

of h_x meets also $E_{1,x}$ at the d_1 distinct points $(0, r_\delta(s_1))$ corresponding to the d_1 distinct roots r_δ of Q defined by :

$$\delta^{d_1} = 1 \quad \text{and} \quad r_\delta(x^{1/d_1}) = r_1(\delta x^{1/d_1}).$$

We find the others intersection points of the strict transform of h_x (by $\pi_{1,x}$) with $E_{1,x}$ with the others roots of Q . The map $\pi_{2,x}$ is the blow-ups of all these intersection points.

Remark 3.4. To make the above blow-ups in family for all $x \in \mathbb{B}_\alpha^2 \setminus \{0\}$, we have to take a sufficiently small α such that:

- (1) $c_{m,0}$ does not vanish on $\mathbb{B}_\alpha^2 \setminus \{0\}$,
- (2) if r and r' are two distinct roots of Q in K_e , then $(r - r')(s^e)$ does not vanish for $s^e \in \mathbb{B}_\alpha^2 \setminus \{0\}$.

End of proof of Lemma 3.3.

As K_{d_1} is nothing but the field of convergent Laurent power series in the variable x^{1/d_1} , there exists $l_1 \in \mathbb{N}^*$ such that

$$x^{l_1} r_1(x^{1/d_1}) \in \mathbb{C}\{x^{1/d_1}\}$$

We consider new local coordinates $(\tilde{u}_1, \tilde{v}_1)$ in $Y_{1,x}$ centered at $(0, r_1(x^{1/d_1}))$ by setting :

$$(u_1, v_1) = (x^{l_1} \tilde{u}_1, \tilde{v}_1 + r_1(x^{1/d_1})) \quad (**)$$

We then have :

$$\pi_{1,x}(\tilde{u}_1, \tilde{v}_1) = (x, u_1, u_1 v_1) = (x, x^{l_1} \tilde{u}_1, (x^{l_1} \tilde{u}_1)(\tilde{v}_1 + r_1(x^{1/d_1})))$$

As $x^{l_1} \tilde{u}_1$ and $(x^{l_1} \tilde{u}_1)(\tilde{v}_1 + r_1(x^{1/d_1}))$ are in $\mathbb{C}\{x^{1/d_1}\}[\tilde{u}_1, \tilde{v}_1]$ and as $\tilde{u}_1 = 0$ is the local equation of $E_{1,x}$ at the point P_1 , statements (1) and (2) of lemma (3.3) are proved for $\pi_{1,x}$.

When we perform $\pi_{2,x}$, we blow-up P_1 in $Y_{1,x}$. Then, $\pi_{2,x}$ can be written in one of the two standard charts by substituting $(\tilde{u}_1, \tilde{v}_1) = (u_2, u_2 v_2)$ or $(\tilde{u}_1, \tilde{v}_1) = (u_2 v_2, v_2)$ in (*). If necessary, we follow it by a new change of coordinates of the type:

$$(u_2, v_2) = (x^{l_2} \tilde{u}_2, \tilde{v}_2 + r_2(x^{1/d_2})),$$

where $x^{l_2} r_2(x^{1/d_2}) \in \mathbb{C}\{x^{1/d_2}\}$ is defined as before.

Then, points 1. and 2. of lemma (3.3) are also proved for $\pi_{2,x} \circ \pi_{1,x}$. By finite iteration, there are also proved for π_x . As π_x is a good resolution of h_x , the strict transform \tilde{h}_x is transverse to E_x at P and has multiplicity 1. A direct computation of $h_x \circ \pi_x$, with the help of the point 2. of lemma (3.3) implies point 3.

This ends the proof of lemma (3.3). \square

Let $\mathcal{U}(\alpha)$ be the interior of $(\mathbb{B}_{2\alpha}^2 \setminus \{0\})$ and $\mathcal{H} = H \cap (\mathcal{U}(\alpha) \times \mathbb{B}_\epsilon^4)$. Let π_1 be the blow-up of the one-dimensional non singular open analytic subset $(\mathcal{U}(\alpha) \times 0 \times 0)$ in \mathbb{C}^3 .

$$\pi_1 : Y_1 \rightarrow (\mathcal{U}(\alpha) \times \mathbb{B}_\epsilon^4).$$

Remark 3.5. For all $x \in \mathcal{U}(\alpha)$, $\pi_{1,x}$, the blow-up of $(x, 0, 0)$ in $\{x\} \times \mathbb{C}^2$, is equal to π_1 restricted on $(\pi_1^{-1}(\{x\} \times \mathbb{B}_\epsilon^4))$. Moreover, for a sufficiently small α , π_x , the minimal good resolution of h_x (see 3.3) is the composition of the same number, let say k , of blow-ups of points.

Let \mathcal{H}_1 be the strict transform (by π_1) of \mathcal{H} . If 2α satisfies the two conditions given in (3.4), \mathcal{H}_1 meets the exceptional divisor $E_1 = \pi_1^{-1}(\mathcal{U}(\alpha) \times 0 \times 0)$ along a one-dimensional non singular open analytic subset of Y_1 . More precisely, in the chart (u_1, v_1) used in the proof of lemma 3.3, the connected components of $E_1 \cap \mathcal{H}_1$ are parametrized by $\{(s, 0, r(s)), s^e \in \mathcal{U}(\alpha)\}$ for all roots $r \in K_e$ of Q . Then, for a sufficiently small α , the open set $E_1 \cap \mathcal{H}_1$ is non singular. Let π_2 be the blow-up of $E_1 \cap \mathcal{H}_1$ in Y_1 . We iterate the same process to obtain $\pi = \pi_k \circ \dots \circ \pi_2 \circ \pi_1$ where

$$\pi : Y_k \rightarrow (\mathcal{U}(\alpha) \times \mathbb{B}_\epsilon^4).$$

By construction, for each $x \in \mathcal{U}(\alpha)$, the restriction of π on $\pi^{-1}(\{x\} \times \mathbb{B}_\epsilon^4)$ is equal to the minimal good resolution π_x of h_x . It is why we say that π is a resolution in family of h_x for $x \in \mathcal{U}(\alpha)$.

Let \mathcal{H}_k be the strict transform of \mathcal{H} by π .

Lemma 3.6. *Each connected component of $(\pi^{-1}(\mathcal{U}(\alpha) \times 0 \times 0)) \cap \mathcal{H}_k$ has an open neighborhood parametrized in s, u and v such that there exist a positive integer M , $c(s) \in (\mathbb{C}\{s\} \setminus \{0\})$ and $g(s, u, v) \in \mathbb{C}\{s, u, v\}$ which satisfy :*

$$(h \circ \pi)(s, u, v) = u^M (ug(s, u, v) + c(s)v).$$

Proof. Let $\mathcal{H}^{(1)}$ be the connected component of \mathcal{H}_k which contains the strict transform $\tilde{h}_{1,x}$ considered in lemma 3.3. Point (3) of lemma 3.3 implies that for all $x \in \mathcal{U}(\alpha)$, we can trivially parametrized by s , $s^d = x$, the same chart in (u, v) . This chart contains $(\pi^{-1}(\mathcal{U}(\alpha) \times 0 \times 0)) \cap \mathcal{H}^{(1)}$. Lemma 3.3 gives the number M , and the series $c(s)$ and $g(s, u, v)$.

This ends the proof of lemma (3.6). \square

Remark 3.7. By definition $(\pi(\mathcal{H}^{(1)})) \cap (\mathbb{S}_\alpha^1 \times \mathbb{B}_\epsilon^4)$ is a branch G of H .

Then $G = \pi(\tilde{G})$ where:

$$\tilde{G} = \{ug(s, u, v) + c(s)v = 0, \ s^d \in \mathbb{S}_\alpha^1, (u, v) \in \mathbb{B}_\epsilon^4\}.$$

Proof of Theorem 3.2.

Thanks to lemma (3.6) and the above remark we have to solve the following equation:

$$\{ug(s, u, v) + c(s)v = 0\},$$

where $c(s) \neq 0$.

Let us perform the change of coordinate $u' = (c(s))^{-1}u$. Then, we obtain :

$$(h \circ \pi)(s, u', v) = u'^M c(s)^{M-1} (u' g(s, u'c(s), v) + v)$$

We replace u' by u . Now the equation of \tilde{G} is given by:

$$u g(s, u, v) + v = 0$$

Let us consider $F(u, v) = u g(s, u, v) + v = 0$ as an element of $A\{u, v\}$ where $A = \mathbb{C}\{s\}$. As $F(0, v) = v$, we can applied the Weierstrass preparation theorem (for example see [23], vol.2, p.139-141), to obtain $R(s, u) \in \mathbb{C}\{s\}\{u\}$ such that

$$F(u, v) = 0 \Leftrightarrow v = R(s, u)$$

This leads to :

$$h \circ \pi(s, u, R(s, u)) = 0.$$

This equality, together with point 2. of lemma (3.3), implies that h vanishes on $\{(s^d, u^i \phi(s, u, R(s, u)), u^j \psi(s, u, R(s, u))), u \in \mathbb{B}_\epsilon^2\}$.

For each $s \in S_{\alpha^{1/d}}^1$, we set $b(s, u) = \phi(s, u, R(s, u))$ and $c(s, u) = \psi(s, u, R(s, u))$. We have a parametrization

$$(S_{\alpha^{1/d}}^1) \times \mathbb{B}_\epsilon^2 \rightarrow G$$

given by

$$(s, u) \longmapsto (s^d, u^i b(s, u), u^j c(s, u))$$

This ends the proof of theorem 3.2. \square

4. M_t IS WALDHAUSEN : THE PROOF

The aim of this section is to prove the main result of this paper :

Theorem 4.1. *M_t is a Waldhausen manifold.*

According to Theorem 2.14, we have to prove that for each branch σ of the singular locus $\Sigma(f)$, the vanishing zone $M_t(\sigma)$ of L_t along σ is a Waldhausen manifold.

4.2. Abstract of the proof

Before giving the proof in details, let us give the key ideas and steps.

- At first, we will show that it suffices to prove that $M_t(\sigma)$ is Waldhausen when σ is smooth. We will then assume that σ is the x -axis.
- Let $\Psi : (\mathbb{C}^3, 0) \rightarrow (\mathbb{C}^3, 0)$ be the germ defined by $\Psi(x, y, z) = (x, y, f(x, y, z))$. The critical locus of Ψ is $H = \{\frac{\partial f}{\partial z} = 0\}$, and its discriminant locus is the image $H' = \Psi(H)$. Let Ψ_t be the restriction of Ψ on $M_t(\sigma)$. As σ is the x -axis, the image of Ψ_t is equal to $\mathbb{S}_\alpha^1 \times \mathbb{B}_\theta^2 \times \{t\}$. Moreover,

$$\Psi_t : M_t(\sigma) \rightarrow \mathbb{S}_\alpha^1 \times \mathbb{B}_\theta^2 \times \{t\},$$

is a finite ramified cover over the solid torus $T = \mathbb{S}_\alpha^1 \times \mathbb{B}_\theta^2 \times \{t\}$ whose ramification locus is the braid $H_t = M_t(\sigma) \cap H$.

Set $H'_t = \Psi_t(H_t)$. To describe H'_t , we consider a branch G of H along the circle $\mathbb{S}_\alpha^1 \times \{0\} \times \{0\}$ as defined in Section 3, we set $G' = \Psi(G)$, and we prove the following parametrization result (Lemma (4.6)) :

if $(x, y, t) \in G'$, then y satisfies the following equality :

$$y = b w(x^{1/d}) x^{e/d'} t^{q/p} + \sum_{m=1}^{\infty} b_m(x^{1/n'}) t^{r_m}, \quad (*)$$

where $b \in \mathbb{C}^*$, d, d', n, p, p' and q are positive integers with $pp' = n$, $w(x^{1/d}) = 1 + \sum_{m=1}^{\infty} w_m x^{m/d} \in \mathbb{C}\{x^{1/d}\}$, $e \in \mathbb{Z}$, $n' = dn$, $b_j(x^{1/n'}) \in K_{n'}$ and $r_m = (qp' + m)/pp'$.

The equality $(*)$ implies that $G'_t = \Psi_t(G \cap M_t(\sigma))$ is a braid in the solid torus T . But this braid can be rather complicated. It is the reason why we approximate it by the torus link

$$\text{App}(G'_t) = \{(x, b x^{e/d'} t^{q/p}, t); x \in \mathbb{S}_\alpha^1\}.$$

Definition. We say that G'_t is the braid of $G' = \Psi(G)$, that $App(G'_t)$ is the torus link associated to G' and that the pair $(q/p, e/d')$ is the pair of the first exponents of G' .

- We index the pairs of first exponents $(q/p, e/d')$ by lexicographic order. For each of them, $(q_i/p_i, e_{i,j}/d'_{i,j})$ where $1 \leq i \leq k$, and $1 \leq j \leq l_i$, we construct a vertical polar zone $Z_{(i,j)}$ (see Definition (4.9)) such that G'_t is included in the interior of $Z_{(i,j)}$ if and only if $(q_i/p_i, e_{i,j}/d'_{i,j})$ is the pair of the first exponents of G' (Lemma (4.10)). Moreover, $Z_{(1,1)}$ is a solid torus and for all (i,j) not equal to $(1,1)$, the $Z_{(i,j)}$'s are concentric thickened tori which recover the solid torus T along their boundaries.
- In the solid torus T , we define some tubular neighborhoods $\mathcal{N}(G'_t)$ of the link $App(G'_t)$ for all the branches G of H such that :
 - * $G'_t \subset \mathcal{N}(G'_t)$
 - * If G has its pair of first exponents indexed by (i,j) , then $\mathcal{N}(G'_t)$ is included in the interior of $Z_{(i,j)}$.
 - * Let \tilde{G} be another branch of H . If $App(G'_t) = App(\tilde{G}'_t)$, then $\mathcal{N}(G'_t) = \mathcal{N}(\tilde{G}'_t)$. Otherwise, $\mathcal{N}(G'_t)$ and $\mathcal{N}(\tilde{G}'_t)$ are disjoint solid tori in T (see Lemma (4.12)).

We call the solid tori $\mathcal{N}(G'_t)$ the *approximation tori*.

Notation. Let $\mathcal{N}(i,j)$ be the union of all the approximation tori of the branches which have their first exponents indexed by (i,j) .

By construction the closure of $Z_{(i,j)} \setminus \mathcal{N}_{(i,j)}$ does not meet the set of ramification values H'_t of Ψ_t and is saturated by $(e_{i,j}, d'_{i,j})$ torus links. The case $e_{i,j} = 0$ is not excluded, but we always have $0 < d'_{i,j}$. It induces a Seifert structure on the closure of $\Psi_t^{-1}(Z_{(i,j)} \setminus \mathcal{N}_{(i,j)})$.

- The last step consists in showing (see Lemma (4.18)) that $\Psi_t^{-1}(\mathcal{N}_{(i,j)})$ is a disjoint union of solid tori. Then it allow us to extend the Seifert fibration on all the $\Psi_t^{-1}(Z_{(i,j)})$. It ends the proof of Theorem 4.1. To prove 4.18, we need Lemma (4.14) which uses deeply the polar curve theory and the Lê-swing theorem (introduced by D.T.Lê and B.Perron in [11]) via the following construction :

Carrousel in family

Let $M(\eta, \sigma)$ (as defined in 2.13), be the union of the $M_t(\sigma)$ where $t \in \mathbb{B}_\eta^2$. The image of the restriction of Ψ on $M(\eta, \sigma)$ is equal to $\mathbb{S}_\alpha^1 \times \mathbb{B}_\theta^2 \times \mathbb{B}_\eta^2$.

Let us fix $a \in \mathbb{S}_\alpha^1$ and let us consider the plane curve germ f_a ,

$$f_a : (\{a\} \times \mathbb{C}^2, (a, 0, 0)) \rightarrow (\mathbb{C}, 0)$$

defined by $f_a(y, z) = f(a, y, z)$. The restriction of Ψ on $M(\eta, \sigma) \cap \{x = a\}$ has $\Gamma_a = H \cap \{x = a\}$ as singular locus. The curve Γ_a is nothing but the polar curve (at $(a, 0, 0)$) of f_a for the direction y , and the set $\Delta_a = \Psi(\Gamma_a)$ is its discriminant curve.

Let us consider

$$M^{(a)}(\sigma) = M(\eta, \sigma) \cap \{x = a\} \cap \{|f| = \eta\}.$$

By construction, the restriction

$$\Psi^{(a)} : M^{(a)}(\sigma) \rightarrow \{a\} \times \mathbb{B}_\theta^2 \times \mathbb{S}_\eta^1$$

of Ψ on $M^{(a)}(\sigma)$ is a ramified cover, whose ramification locus is $\Gamma_a \cap \{|f| = \eta\}$.

Remark. By construction, the Milnor fiber of the plane curve germ f_a is

$$F_{t,a} = M_t(\sigma) \cap M^{(a)}(\sigma).$$

The restriction $\psi_a : F_{t,a} \rightarrow \mathcal{D}$ of $\Psi^{(a)}$ on $F_{t,a}$ is a finite ramified cover over the disk $\mathcal{D} = \{a\} \times \mathbb{B}_\theta^2 \times \{t\}$. This ramified cover has been studied in details by D.T.Lê, (for example in [9] and in [10]) to study the monodromy of the Milnor fiber as a pull-back (here by ψ_a) of a diffeomorphism of the disk \mathcal{D} modulo its intersection points with Δ_a . D.T.Lê calls this construction “the carrousel”.

But ψ_a is also the restriction of Ψ_t on $F_{t,a}$. Then we have to study the family of ψ_x with $x \in \mathbb{S}_\alpha^1$. In order to do this, we construct a carrousel parametrized by x : it is a carrousel in family.

This ends the abstract of the proof. In the rest of this section, we provide detailed proofs.

4.3. Reduction to a smooth branch of Γ_t

Let us fix a branch σ of $\Sigma(f)$ and let

$$u \mapsto (u^k, \phi(u), \psi(u))$$

be a Puiseux parametrization of σ .

Let us consider the analytic morphism $\Theta : \mathbb{C}^3 \rightarrow \mathbb{C}^3$ defined by

$$\Theta(x, y, z) = (x^k, y + \phi(x), z + \psi(x))$$

Let $g : (\mathbb{C}^3, 0) \rightarrow (\mathbb{C}, 0)$ be the composition $g = f \circ \Theta$. Then $\sigma' = \Theta^{-1}(\sigma)$ is the x -axis. Moreover, a direct computation of the derivative of g shows that σ' is a branch of the singular locus of g .

Let $M_t(f, \sigma)$ (resp. $M_t(g, \sigma')$) be the vanishing zone of f along σ (resp. of g along σ') defined in the boundary of the ball $B(\alpha)$ (resp. $B(\alpha^{1/k})$) as in 2.13. The construction given in 2.13 leads directly to :

Lemma 4.4. *$M_t(g, \sigma) = \Theta^{-1}(M_t(f, \sigma))$, and the restriction $\Theta|_{M_t(g, \sigma')} : M_t(g, \sigma') \rightarrow M_t(f, \sigma)$ is a diffeomorphism.*

In the sequel, we assume that σ is the x -axis. In particular, the vanishing zone $M_t(\sigma)$ along σ is nothing but

$$M_t(\sigma) = L_t \cap (\mathbb{S}_\alpha^1 \times \mathbb{B}_\theta^2 \times \mathbb{B}_\gamma^2), \quad 0 < \eta < \theta < \alpha.$$

4.5. Parametrization of the branches of $\Psi(H)$

Let us recall that $\Psi : (\mathbb{C}^3, 0) \rightarrow (\mathbb{C}, 0)$ denotes the germ defined by $\Psi(x, y, z) = (x, y, f(x, y, z))$. The critical locus of Ψ is $H = \{\frac{\partial f}{\partial z} = 0\}$, and its discriminant locus is the image $H' = \Psi(H)$.

Let G be the closure (in \mathbb{C}^3) of a connected component of

$$(H \setminus (\mathbb{S}_\alpha^1 \times \{0\} \times \{0\})) \cap (\mathbb{S}_\alpha^1 \times \mathbb{B}_\theta^2 \times \mathbb{B}_\gamma^2),$$

i.e. G is a branch of H along the circle $\mathbb{S}_\alpha^1 \times \{0\} \times \{0\}$ as defined in Section 3.

We set $G' = \Psi(G)$, and we call G' a branch of $H' = \Psi(H)$ along $\mathbb{S}_\alpha^1 \times \{0\} \times \{0\}$.

Let us recall that K_d denotes the fraction field of $\mathbb{C}\{x^{1/d}\}$.

Lemma 4.6. *There exist:*

- $d, n, p, p', q \in \mathbb{N}^*$, where p is prime to q and $pp' = n$,
- $e \in \mathbb{Z}$ and $d' \in \mathbb{N}^*$ is prime to e (if $e = 0$, then $d' = 1$),
- $r_m = (qp' + m)/pp'$,
- $b_j(x^{1/n'}) \in K_{n'}$, where $n' = dn$,
- $w(x^{1/d}) = 1 + \sum_{m=1}^{\infty} w_m x^{m/d} \in \mathbb{C}\{x^{1/d}\}$ and $b \in \mathbb{C}^*$.

such that, if $(x, y, t) \in G'$, then y satisfies the following equality :

$$y = b w(x^{1/d}) x^{e/d'} t^{q/p} + \sum_{m=1}^{\infty} b_m(x^{1/n'}) t^{r_m} \quad (*)$$

Remark. As recalled below, the integer d is provided by theorem 3.2 . For each branch G of H there exists such a d minimal which depends on G . Here, for convenience, we will choose a (perhaps greater) d common to all the branches of H .

Proof. Theorem 3.2 provides $b(x^{1/d}, u) \in \mathbb{C}\{x^{1/d}\}\{u\}$ and $c(x^{1/d}, u) \in \mathbb{C}\{x^{1/d}\}\{u\}$ with $b(x, 0) \neq 0$ and $c(x, 0) \neq 0$, such that we have a parametrization

$$\mathbb{S}_{\alpha^{1/d}}^1 \times \mathbb{B}_\epsilon^2 \rightarrow G$$

given by

$$(s, u) \mapsto (s^d, u^i b(s, u), u^j c(s, u))$$

and we obtain $n, j < n$, and $c'(x^{1/d}, u) \in \mathbb{C}\{x^{1/d}\}\{u\}$ with $c'(x, 0) \neq 0$, such that $G' = \Psi(G)$ admits a parametrization of the form

$$(s, u) \mapsto (s^d, u^i b(s, u), u^n c'(s, u)) \quad (**)$$

If necessary, we can perform the modification $u = s^{l'} u', l' \in \mathbb{N}$, to obtain $l \in \mathbb{N}$, $c_m(s) \in \mathbb{C}\{s\}$ with $c_0(0) \in \mathbb{C}^*$, such that:

$$t = u^n c'(u, s) = u'^n s^l c_0(s) \left(1 + \sum_{m=1}^{\infty} c_m(s) u'^m \right)$$

There then exist $r(x^{1/d}, u') \in \mathbb{C}\{x^{1/d}\}\{u'\}$ with $r(0, 0) = 1$ and $r_0(x^{1/d}) \in \mathbb{C}\{x^{1/d}\}$ with $(r_0(0))^n = c_0(0)$, such that

$$t = u'^n s^l (r_0(s))^n (r(s, u'))^n.$$

We perform the following change of coordinates:

$$u_1 = u' r_0(s) r(s, u')$$

and $(**)$ becomes:

$$(s, u_1) \mapsto (s^d, u_1^i b'(s, u_1), u_1^n s^l)$$

where $b'(x^{1/d}, u_1) \in \mathbb{C}\{x^{1/d}\}\{u_1\}$.

Now $u_1 = s^{-l/n} t^{1/n}$ and $(x, y, t) \in G'$ satisfies:

$$y = (x^{-il/nd} t^{i/n}) b'(x^{1/d}, x^{-l/nd} t^{1/n}) \quad (***)$$

As $x \in \mathbb{S}_\alpha^1$ and $t \in \mathbb{B}_\eta^2$ with $0 < \eta \ll \alpha$, there is no problem of convergency. Moreover, we have :

$$b'(x^{1/d}, 0) = b(x^{1/d}, 0) = b x^{k/d} (1 + \sum_{m=1}^{\infty} w_m x^{m/d}), k \in \mathbb{N}, b \in \mathbb{C}^*, w_m \in \mathbb{C}.$$

Let

$$w(x^{1/d}) = 1 + \sum_{m=1}^{\infty} w_m x^{m/d} \in \mathbb{C}\{x^{1/d}\},$$

if we take p and q prime to each other such that $q/p = i/n = qp'/pp'$, $n' = nd$, e and d' prime to each other such that $e/d' = (n k - i l)/(nd)$, and if we write $(***)$ in terms of the increasing powers of t , we obtain $(*)$ of Lemma (4.6) i.e. :

$$y = b w(x^{1/d}) x^{e/d'} t^{q/p} + \sum_{m=1}^{\infty} b_m (x^{1/n'}) t^{r_m}.$$

This ends the proof of Lemma 4.6 \square

4.7. The polar decomposition

Let us consider the ordered set

$$Q = \left\{ \frac{q_k}{p_k} < \dots < \frac{q_2}{p_2} < \frac{q_1}{p_1} \right\}$$

of rational numbers $\frac{q}{p}$ such that, there exists a branch G' of $\Psi(H)$ which admits, with the notations of 4.6, a parametrization of the form :

$$y = b w(x^{1/d}) x^{e/d'} t^{q/p} + \sum_{m=1}^{\infty} b_m (x^{1/n'}) t^{r_m},$$

with $x \in \mathbb{S}_\alpha^1$ and $t \in \mathbb{B}_\eta^2$.

We denote by G'_i the union of the branches of $\Psi(H)$ corresponding to the quotient q_i/p_i .

For each $i \in \{1, \dots, k\}$, let

$$Q_i = \left\{ \frac{e_{i,1}}{d'_{i,1}} < \dots < \frac{e_{i,j}}{d'_{i,j}} < \dots < \frac{e_{i,l(i)}}{d'_{i,l(i)}} \right\}$$

be the ordered set of rational numbers such that there exists a branch of G'_i which admits a parametrization of the form:

$$(1) \quad y = b w(x^{1/d}) x^{e_{i,j}/d'_{i,j}} t^{q_i/p_i} + \sum_{m=1}^{\infty} b_m(x^{1/n'}) t^{r_m},$$

with $x \in \mathbb{S}_{\alpha}^1$ and $t \in \mathbb{B}_{\eta}^2$.

We denote by $G'_{i,j}$ the union of such branches of G'_i .

Let us fix $a \in \mathbb{S}_{\alpha}^1$. We consider the plane curve germ $f_a(y, z) = f(a, y, z)$. By definition the above set Q is the set of polar quotients of f_a for the direction y (for example see [9]). We will follow the classical construction of [12] which furnishes a decomposition of the solid torus $T_a = \{a\} \times \mathbb{B}_{\theta}^2 \times \mathbb{S}_{\eta}^1$ into polar zones in bijection with the polar quotients q_i/p_i . This decomposition lifts by $\Psi^{(a)}$ to a Waldhausen decomposition of the exterior of the link of f_a . But as explained in the abstract of the proof, we will in fact define our polar zones Z_i in the solid torus $T = \mathbb{S}_{\alpha}^1 \times \mathbb{B}_{\theta}^2 \times \{t\}$. The key idea is that the two constructions coincide on the disc $\mathcal{D} = T \cap T_a$ where they give a polar decomposition of \mathcal{D} as an union of concentric annuli.

Let us now define this decomposition of T as the union of Z_i .

For each $i \in \{1, \dots, k-1\}$, let us choose $s_i \in \mathbb{Q}$ such that

$$\frac{q_{i+1}}{p_{i+1}} < s_i < \frac{q_i}{p_i},$$

Definition 4.8. The *first polar zone* is the solid torus

$$Z_1 = \{(x, y, t) \in T / |y| \leq \eta^{s_1}\},$$

and $C(1) = Z_1 \cap \mathcal{D}$ is the *first polar disc*.

If $i \in \{2, \dots, k-1\}$, the *polar zone* Z_i is the thickened torus defined by:

$$Z_i = \{(x, y, t) \in T / \eta^{s_{i-1}} \leq |y| \leq \eta^{s_i}\},$$

and $C(i) = Z_i \cap \mathcal{D}$ is the associated *polar annulus*.

In T , the value of $t \in \mathbb{S}_{\eta}^1$ is fixed. If G is a branch of H with first exponents $(q_i/p_i, e_{i,j}/d'_{i,j})$, then the braid $\Psi_t(G) = G'_t$ admits a parametrization of the form (1) in 4.7.

To take account into the first exponent of x , we will refine the polar decomposition of T . For each $j \in \{1, \dots, l_i-1\}$, let us choose a rational number $\nu_{i,j}$ such that

$$\frac{e_{i,j+1}}{d'_{i,j+1}} < \nu_{i,j} < \frac{e_{i,j}}{d'_{i,j}},$$

There exists η sufficiently small, $0 < \eta \ll \theta \ll \alpha$, such that the following inequalities hold :

$$0 < \eta^{q_1/p_1} \alpha^{\nu_{1,1}} < \eta^{q_1/p_1} \alpha^{\nu_{1,2}} < \dots < \eta^{q_1/p_1} \alpha^{\nu_{1,l_1-1}} < \eta^{s_1},$$

for each $i \in \{2, \dots, k-1\}$,

$$\eta^{s_{i-1}} < \eta^{q_i/p_i} \alpha^{\nu_{i,1}} \dots < \eta^{q_i/p_i} \alpha^{\nu_{i,l_i-1}} < \eta^{s_i},$$

and

$$\eta^{s_{k-1}} < \eta^{q_k/p_k} \alpha^{\nu_{k,1}} \dots < \eta^{q_k/p_k} \alpha^{\nu_{k,l_k-1}} < \theta.$$

Definition 4.9. The vertical polar zones $Z_{(i,j)}$, $1 \leq i \leq k$, $1 \leq j \leq l_i$, are defined as follows :

- $Z_{(1,1)}$ is the solid torus

$$Z_{(1,1)} = \{(x, y, t) \in T / |y| \leq \eta^{q_1/p_1} \alpha^{\nu_{1,1}}\},$$

- For (i, j) not equal to $(1, 1)$, $Z_{(i,j)}$ is a thickened torus :
 * If $1 < i \leq k$,

$$Z_{(i,1)} = \{(x, y, t) \in T / \eta^{s_{i-1}} \leq |y| \leq \eta^{q_i/p_i} \alpha^{\nu_{i,1}}\},$$

$$* \text{ if } 1 \leq i \leq k, j = \{2, \dots, l_i - 1\},$$

$$Z_{(i,j)} = \{(x, y, t) \in T / \eta^{q_i/p_i} \alpha^{\nu_{i,j-1}} \leq |y| \leq \eta^{q_i/p_i} \alpha^{\nu_{i,j}}\},$$

$$* \text{ if } 1 \leq i < k,$$

$$Z_{(i,l_i)} = \{(x, y, t) \in T / \eta^{q_i/p_i} \alpha^{\nu_{i,l_i-1}} \leq |y| \leq \eta^{s_i}\},$$

$$* \text{ and}$$

$$Z_{(k,l_k)} = \{(x, y, t) \in T / \eta^{q_k/p_k} \alpha^{\nu_{k,l_k-1}} \leq |y| \leq \theta\}.$$

The associated refined polar annuli are :

$$C(i, j) = Z_{(i,j)} \cap \mathcal{D}$$

By construction the torus T is equal to the union of the vertical polar zones $Z_{(i,j)}$, $1 \leq i \leq k$, $1 \leq j \leq l_i$. The intersection of two consecutive (for the lexicographic order on the (i, j)) vertical polar zones is a unique torus which is the common connected component of their boundaries. The intersection between non consecutive vertical polar zones is empty. But, the most important property of the vertical polar zones is given by Lemma (4.10).

Lemma 4.10. *There exist α and η sufficiently small, $0 < \eta \ll \theta \ll \alpha$, such that a branch G' of $H' = \Psi(H)$ has $(q_i/p_i, e_{i,j}/d'_{i,j})$ as pair of first exponents if and only if the braid $G'_t = \Psi_t(M_t(\sigma) \cap G)$ is included in the interior of $Z_{(i,j)}$.*

Proof. By definition, G' has a parametrization of the form (1) in 4.7:

$$y = b w(x^{1/d}) x^{e_{i,j}/d'_{i,j}} t^{q_i/p_i} + \sum_{m=1}^{\infty} b_m(x^{1/n'}) t^{r_m},$$

Therefore, $(x, y, t) \in G'_t$ if and only if

$$|y| = \alpha^{e_{i,j}/d'_{i,j}} \eta^{q_i/p_i} \left| b w(x^{1/d}) + \sum_{m=1}^{\infty} b_m(x^{1/n'}) t^{r_m - q_i/p_i} (x^{-e_{i,j}/d'_{i,j}}) \right|.$$

Then, the inequality

$$\nu_{i,j} < \frac{e_{i,j}}{d'_{i,j}} < \nu_{i,j-1}$$

implies lemma 4.10 for the zone $Z_{(i,j)}$ where $1 \leq i \leq k$, $j = \{2, \dots, l_i - 1\}$.

As $s_i < \frac{q_i}{p_i} < s_{i-1}$, the computations are similar for the other vertical polar zones.

□

4.11. The approximation solid tori

Let G be a branch of H such that $G' = \Psi(G)$ is parametrized by

$$y = b w(x^{1/d}) x^{e/d'} t^{q/p} + \sum_{m=1}^{\infty} b_m(x^{1/n'}) t^{r_m}.$$

We approximate the braid $G'_t = \Psi_t(G \cap M_t(\sigma))$ by a torus link $App(G'_t)$ as follows :

Definition. The link $App(G'_t)$ associated to the braid G'_t is the torus link in $T = \mathbb{S}^1_\alpha \times \mathbb{B}^2_\theta \times \{t\}$ defined by:

$$App(G'_t) = \{(x, b x^{e/d'} t^{q/p}, t), x \in \mathbb{S}^1_\alpha\}.$$

Let l be the l.c.m. of d' and p . Let $a \in \mathbb{S}^1_\alpha$, let s and τ be such that $s^{d'} = a$ and $\tau^p = t$.

Definition. The *suns* of G'_t are the intersection points $\mathcal{S}(G'_t) = G'_t \cap \mathcal{D} = \{(a, b \xi s^e \tau^q, t), \xi^l = 1\}$

Let $\rho = (e/d' + 1/2d)$.

Definition. We call *approximation solid tori* of G'_t the tubular neighborhood $\mathcal{N}(G'_t)$ of $\text{App}(G'_t)$ defined by:

$$\mathcal{N}(G'_t) = \{(x, y, t) \in T \text{ such that } 0 \leq |y - b x^{e/d'} t^{q/p}| \leq \eta^{q/p} \alpha^\rho\}.$$

Lemma 4.12. *There exist α and η sufficiently small, $0 < \eta \ll \theta \ll \alpha$, such that:*

- (1) *The intersection $\mathcal{N}(G'_t) \cap \mathcal{D}$ consists of l disjoint discs of radius equal to $\eta^{q/p} \alpha^\rho$ which have the l suns of G'_t as centers.*
- (2) *The braid G'_t is included in $\mathcal{N}(G'_t)$.*
- (3) *If $(q_i/p_i, e_{i,j}/d'_{i,j})$ is the pair of the first exponents of G'_t then $\mathcal{N}(G'_t) \subset \text{int}(Z_{(i,j)})$.*
- (4) *Let \tilde{G} be another branch of H . If $\text{App}(G'_t) = \text{App}(\tilde{G}'_t)$, then $\mathcal{N}(G'_t) = \mathcal{N}(\tilde{G}'_t)$. Otherwise, $\mathcal{N}(G'_t)$ and $\mathcal{N}(\tilde{G}'_t)$ are disjoint solid tori in T*

Proof. To obtain (1), it is sufficient to prove that if $\xi \neq 1$, for a sufficiently small α we have:

$$3 \eta^{q/p} \alpha^\rho < |(b - \xi b)| \eta^{q/p} \alpha^{e/d'}.$$

But this inequality is equivalent to:

$$3 \alpha^{1/(2d)} < |(b - \xi b)|.$$

As b is a given non zero complex number, it is sufficient to choose α sufficiently small to obtain (1).

let $(s^{d'}, y, \tau^p) \in G'_t$, then:

$$y = b w(s^{d'/d}) s^e \tau^q + \sum_{m=1}^{\infty} b_m (s^{d'/n'}) \tau^{pr_m}.$$

By construction there exists $w_1(s^{d'/d}) \in \mathbb{C}\{s^{d'/d}\}$ such that:

$$w(s^{d'/d}) - 1 = s^{d'/d} w_1(s^{d'/d}).$$

For sufficiently small, α and η with $0 < \eta \ll \theta \ll \alpha$, we have:

$$|y - b s^e \tau^q| = |b w_1(s^{d'/d}) s^{e+(d'/d)} \tau^q + \sum_{m=1}^{\infty} b_m (s^{d'/n'}) \tau^{pr_m}|$$

$$= \eta^{q/p} \alpha^{e/d'+1/d} |b w_1(s^{d'/d}) + \sum_{m=1}^{\infty} b_m(s^{d'/n'}) s^{(-e - d'/d)} \tau^{-q+pr_m}| < \eta^{q/p} \alpha^{e/d'+1/2d}.$$

We then get (2).

To get (3), we show that, for sufficiently small, α and η with $0 < \eta \ll \theta \ll \alpha$, the distance, in \mathcal{D} , between the suns of G'_t and the two boundary connected components of the annulus $C(i, j)$ is bigger than the radius $\eta^{q/p} \alpha^\rho$.

By construction, we have for $1 \leq i \leq k$ and $j = \{2, \dots, l_i - 1\}$:

$$C(i, j) = \{(a, y, t) \in \mathcal{D} \text{ with } \eta^{q_i/p_i} \alpha^{\nu_{i,j-1}} \leq |y| \leq \eta^{q_i/p_i} \alpha^{\nu_{i,j}}\},$$

where:

$$s_i < \frac{q_i}{p_i} < s_{i-1}, \text{ and } \nu_{i,j} < \frac{e_{i,j}}{d'_{i,j}} < \nu_{i,j-1}.$$

The distance between a sun of G'_t and the interior circle of $C(i, j)$ is equal to:

$$\eta^{q_i/p_i} \alpha^{(e_{i,j}/d'_{i,j})} (|b| - (\alpha^{(\nu_{i,j-1}) - (e_{i,j}/d'_{i,j})})).$$

This distance, for sufficiently small α and η , $0 < \eta \ll \alpha$, is greater than $\eta^{q/p} \alpha^{\rho_{i,j}}$ because the exponent $\rho_{i,j} = (e_{i,j}/d'_{i,j} + 1/2d)$ corresponding to a branch with the pair of the first exponents equal to $(q_i/p_i, e_{i,j}/d'_{i,j})$, is greater than $(e_{i,j}/d'_{i,j})$. But $\nu_{i,j} < \frac{e_{i,j}}{d'_{i,j}} < \rho_{i,j}$, and similar computations prove that the distance between a sun of G'_t and the exterior circle of $C(i, j)$ is bigger than the radius $\eta^{q_i/p_i} \alpha^{\rho_{i,j}}$.

Then (3) is done.

Let us now prove (4). When $App(G'_t) = App(\tilde{G}'_t)$, then by definition $\mathcal{N}(G'_t) = \mathcal{N}(\tilde{G}'_t)$.

If \tilde{G}'_t does not have the same pair of first exponents as G'_t then $\mathcal{N}(G'_t)$ and $\mathcal{N}(\tilde{G}'_t)$ are included in the interior of distinct vertical polar zones, they do not meet.

The last case is when G'_t and \tilde{G}'_t have the same pair of first exponents $(q/p, e/d')$, but distinct associated torus link. If $(s^{d'}, y, \tau^p) \in G'_t$, then:

$$y = b w(s^{d'/d}) s^e \tau^q + \sum_{m=1}^{\infty} b_m(s^{d'/n'}) \tau^{pr_m}.$$

If $(s^{d'}, y, \tau^p) \in \tilde{G}'_t$, then:

$$y = \tilde{b} \tilde{w}(s^{d'/d}) s^e \tau^q + \sum_{m=1}^{\infty} \tilde{b}_m(s^{d'/n'}) \tau^{pr_m},$$

where $\tilde{b} \in \mathbb{C}^*$ and $\tilde{b} \neq \xi b$, for all ξ such that $\xi^l = 1$. But the minimal value of $\{|\tilde{b} - \xi b|, \xi^l = 1\}$ is well defined. With computations similar of

those performed to obtain points (1) and (2), we can choose sufficiently small α and η , $0 < \eta \ll \alpha$, such that the distances between the suns of G'_t and \tilde{G}'_t are bigger than $3\eta^{q/p}\alpha^\rho$. This proves that $\mathcal{N}(\tilde{G}'_t)$ and $\mathcal{N}(G'_t)$ are disjoint.

But the trivial projection of $T = \mathbb{S}_\alpha^1 \times \mathbb{B}_\theta^2 \times \{t\}$ on \mathbb{S}_α^1 restricted on $\mathcal{N}(G'_t)$ is a fibration with the discs $\mathcal{N}(G'_t) \cap \mathcal{D}$ as fiber. Then the tubular neighborhoods $\mathcal{N}(G'_t)$ are an union of disjoint solid tori in T

This ends the proof of lemma 4.12 \square

Lemma (4.12) allows us to define the solar discs.

Definition 4.13. Let s and τ be such that $s^{d'} = a$ and $\tau^p = t$. If G is a branch of H and $G' = \Psi(G)$ the solar discs associated to G are the l disjoint discs $\mathcal{N}(G'_t) \cap \mathcal{D}$ centered at the suns $\mathcal{S}(G'_t) = G'_t \cap \mathcal{D} = \{(a, b \xi s^e \tau^q, t), \xi^l = 1\}$ of G'_t .

Lemma 4.14. Let D_G be a solar disc of G , then $\Psi_t^{-1}(D_G)$ is a disjoint union of discs.

To prove lemma 4.14 we need the following subsection.

4.15. Carrousel in family

Let $M(\eta, \sigma)$ (as defined in 2.13), be the union of the $M_t(\sigma)$ where $t \in \mathbb{B}_\eta^2$. The image of the restriction of Ψ on $M(\eta, \sigma)$ is equal to $\mathbb{S}_\alpha^1 \times \mathbb{B}_\theta^2 \times \mathbb{B}_\eta^2$.

Let us fix $a \in \mathbb{S}_\alpha^1$ and let us consider the plane curve germ f_a ,

$$f_a : (\{a\} \times \mathbb{C}^2, (a, 0, 0)) \rightarrow (\mathbb{C}, 0)$$

defined by $f_a(y, z) = f(a, y, z)$. The restriction of Ψ on $(M(\eta, \sigma) \cap \{x = a\})$ has $\Gamma_a = H \cap \{x = a\}$ as singular locus, it is the polar curve (at $(a, 0, 0)$) of f_a for the direction y . The set $\Delta_a = \Psi(\Gamma_a)$ of its singular values is the corresponding discriminant curve.

By construction, the Minor fiber of the plane curve germ f_a is

$$F_{t,a} = M_t(\sigma) \cap \{x = a\}.$$

Let ψ_a be the restriction of Ψ on $F_{t,a}$:

$$\psi_a : F_{t,a} \rightarrow \mathcal{D}.$$

As ψ_a is equal to the restriction of Ψ_t on $F_{t,a}$. Lemma 4.14 is equivalent to:

Claim. Let D_G be a solar disc of G , then $\psi_a^{-1}(D_G)$ is a disjoint union of discs.

Now we will prove this claim.

Let δ be an irreducible component of the discriminant Δ_a which is included in $G' = \Psi(G)$. Then a Puiseux expansion of δ is given by:

$$y = b w(s^n) s^{ed''} t^{q/p} + \sum_{m=1}^{\infty} b_m(s) t^{r_m}.$$

Where s and d'' satisfy the following equalities: $s^{nd} = a$ and $d'd'' = nd$. Moreover, the suns of δ as defined in [12], in (2.4.3)p.157, are the following p points of \mathcal{D} : $\{(a, b w(s^n) s^{ed''} \tau^q, t), \tau^p = t\}$. In [12], a solar "polar" disc D is defined in (2.4.6), and lemma 2.4.7 states that $\psi_a^{-1}(D)$ is a disjoint union of discs. This uses the L \hat{e} -swing. Our polar disc D_G takes account of the coefficients parametrized by x via $w(x^{1/d})$ and is slightly different from D . But we can consider the curve δ' having $y = b s^{ed''} t^{q/p}$ as Puiseux expansion in $\{a\} \times \mathbb{C}^2$. If we use the curve δ' in the proof of lemma (2.4.7) (in [12],) in place of δ_0 , we obtain, with exactly the same arguments, that $\psi_a^{-1}(D_G)$ is a disjoint union of discs. This proves the claim. \square

Remark 4.16. In [2], C.Caubel proves a very general version of the L \hat{e} -swing. In particular let D be a subdisc of a polar annuli $C(i, j)$. We say that D is marked if it contains points of Δ_a in its interior, but the boundary of D does not meet Δ_a . Proposition (2.4) in [2], implies that:

if D is a marked subdisc contained in a sector, in $C(i, j)$, of angle θ with $\theta < 2\pi(q_i/p_i + 1/2p_i)$, then D can be swung.

Then, in the case a plane curve germ (as f_a in our case), we obtained (as proved in (2.4.12) of [12]), that $\psi_a^{-1}(D)$ is a disjoint union of discs. By definition our polar disc D_G is contained in such a sector.

4.17. Vertical monodromy

Let p be the restriction on $M_t(\sigma)$ of the projection on the x -axis i.e.:

$$p : M_t(\sigma) \rightarrow \mathbb{S}_\alpha^1.$$

In (2.1) we choose a generic x -axis such that p is a submersion on $M_t(\sigma)$ when $t \in \mathbb{S}_\eta^1$, $0 < \eta \ll \alpha$. Then p is a differentiable fibration of fiber $F_{t,a}$ and $M_t(\sigma)$ is the mapping-torus of a diffeomorphism $h : F_{t,a} \rightarrow F_{t,a}$ of p . Following the terminology introduced by D. Siersma in [20], h is a *vertical monodromy* for σ .

Let $\mathcal{N}(i, j)$ be the union of all the approximation tori of the branches which have their first exponents indexed by (i, j) .

Lemma 4.18. *Each $\Psi_t^{-1}(\mathcal{N}_{(i,j)})$ is a disjoint union of solid tori.*

Proof of Lemma 4.18. By construction the boundary of $\Psi_t^{-1}(\mathcal{N}_{(i,j)})$ meets $\{x = a\}$ transversally for all $a \in \mathbb{S}_\alpha^1$. Then, the restriction $p_{i,j}$ of p on $\Psi_t^{-1}(\mathcal{N}_{(i,j)})$ is a fibration. But the fibers of this restriction is a disjoint union of $\Psi_t^{-1}(D_G)$ for all the polar discs D_G of the branches $G' = \Psi(G)$ having $(q_i/p_i, e_{i,j}/d'_{i,j})$ as pair of first exponents. Lemma 4.14 implies that the fibers of $p_{i,j}$ are a disjoint union of discs. Then $\Psi_t^{-1}(\mathcal{N}_{(i,j)})$ is the mapping torus of a disjoint union of discs, it is a disjoint union of solid tori.

□

Lemma 4.18 is the key-lemma which enables one to conclude :

By construction the closure of $Z_{(i,j)} \setminus \mathcal{N}_{(i,j)}$ does not meet the ramification value H'_t of Ψ_t and is saturated by $(e_{i,j}, d'_{i,j})$ torus links. The case $e_{i,j} = 0$ is not excluded, but we always have $0 < d'_{i,j}$. It induces a Seifert structure on the closure of $\Psi_t^{-1}(Z_{(i,j)} \setminus \mathcal{N}_{(i,j)})$. Moreover the so obtained Seifert leaves are, by construction, transverse to the fibers of p . Then, lemma 4.18 allows us to extend the Seifertic structure on the disjoint union of solid tori $\Psi_t^{-1}(\mathcal{N}_{(i,j)})$, the connected components of $\Psi_t^{-1}(D_G)$ being the meridian discs of the tori $\Psi_t^{-1}(\mathcal{N}_{(i,j)})$, there is no singular leaf in the constructed Seifert structure on $\Psi_t^{-1}(Z_{(i,j)})$ and the possible exceptional leaves are the cores of the tori $\Psi_t^{-1}(\mathcal{N}_{(i,j)})$ or in $\Psi_t^{-1}(\mathbb{S}_\alpha^1 \times \{0\} \times \{t\})$. The union along their boundaries of the Seifert manifolds $\Psi_t^{-1}(Z_{(i,j)})$, for all (i, j) gives a waldhausen structure on $M_t(\sigma) = \Psi_t^{-1}(\mathbb{S}_\alpha^1 \times \mathbb{B}_\theta^2 \times \{t\})$. This ends the proof of Theorem 4.1

□

5. A TOPOLOGICAL CHARACTERIZATION OF ISOLATED SINGULARITIES

In this section, we prove the following topological characterization of isolated singularities, which was the first motivation of this work.

Theorem 5.1. *Let $f : (\mathbf{C}^3, 0) \longrightarrow (\mathbf{C}, 0)$ be a reduced holomorphic germ whose singular locus is at most 1-dimensional. We assume that either f is reducible or L_t is not a lens space. Then the following assertions are equivalent.*

- (i) *f is either smooth or has an isolated singularity at 0.*
- (ii) *The boundary L_t , $t \neq 0$ of the Milnor fibre of f is homeomorphic to the link $\overline{L_0}$ of the normalization of $f^{-1}(0)$.*

The degenerating case when f is irreducible and L_t is a lens space remains open. The following proposition shows that, in fact, this case

concerns a very special family of singularities. Recall that the $\overline{K_0}$ denotes the link of the curve $\overline{\Sigma(f)} = n^{-1}(\Sigma(f))$ in the link $\overline{L_0}$ of the normalization $\overline{F_0}$ of F_0 .

Proposition 5.2. *Let $f : (\mathbb{C}^3, 0) \longrightarrow (\mathbb{C}, 0)$ be a reduced holomorphic germ such that f is irreducible and L_t is a lens space. Then*

- (1) *The trunk N_0 is a solid torus, $\overline{L_0}$ is a lens space, $\overline{\Sigma(f)}$ is an irreducible germ of curve and the resolution graph of the pair $(\overline{F_0}, \overline{\Sigma(f)})$ is a bamboo with an arrow at one of its extremities,*
- (2) *M_t is connected with a connected boundary.*

Proof of proposition 5.2. Let σ be a component of $\Sigma(f)$. According to 4.17, $M_t(\sigma)$ is fibred over the circle with fiber $F_{t,a}$. As $F_{t,a}$ is not a disk, then $M_t(\sigma)$ is not a solid torus.

Let T be a connected component of $\partial N_0 = \partial M_t$. As the connected components of M_t are irreducible manifolds (5.3) none of them being a solid torus, then T is incompressible in M_t (see [15], 9.1, prop. D). Now, as the trunk N_0 is irreducible (5.3), if it were not a solid torus, T would also be incompressible in N_0 (see again [15], 9.1, prop. D). Then, van Kampen's Theorem and Dehn's Lemma would imply that T is incompressible in L_t . But a torus embedded in a lens space is always compressible. Hence N_0 is a solid torus. It follows immediately that $\overline{L_0}$ is a lens space. According to 2.8, the curve $\overline{\Sigma(f)}$ is irreducible in $\overline{F_0}$. Therefore $\Sigma(f)$ is also irreducible.

As the trunk is a solid torus, the vanishing zone M_t is connected with a connected boundary because $\partial N_0 = \partial M_t$.

□

The proof of Theorem 5.1 needs the following :

Lemma 5.3. *The trunk $\overline{N_0}$ and the vanishing zone M_t are irreducible 3-manifolds.*

Recall that a 3-manifold M is irreducible if every embedded 2-sphere in M is the boundary of a 3-ball.

Proof of Lemma 5.3. It suffices to prove that every connected component \overline{W} of $\overline{N_0}$ is irreducible. Let (S, p) be an irreducible component of $\overline{F_0}$ whose link contains \overline{W} , and set $\gamma = \overline{\Sigma(f)} \cap S$. Then \overline{W} is the complement of a tubular neighborhood of the link of the complex germ of curve (γ, p) in the link of the normal complex surface singularity (S, p) . Therefore \overline{W} is irreducible (see [15], 9.2, Cor. J).

According to 4.17, each connected component $M_t(\sigma)$ of the vanishing zone M_t is fibered over the circle \mathbb{S}^1 with a connected and orientable fibre not diffeomorphic to the 2-sphere. Therefore $M_t(\sigma)$ is irreducible (see [15], 9.1., Lemma A).

Proof of theorem 5.1 (i) \Rightarrow (ii) follows from Milnor's theory ([16]).

To prove (ii) \Rightarrow (i), let us assume that f is neither smooth nor has an isolated singularity at 0. When f is not irreducible, then $\overline{L_0}$ has several connected components whereas L_t is connected (Corollary 2.15). Then we assume that the germ f is irreducible.

If the trunk $\overline{N_0} \cong N_t$ is a solid torus, then $\overline{L_0}$ is a lens space. Then as we have assumed that L_t is not a lens space, then L_t is not homeomorphic to $\overline{L_0}$.

Now, assume that the trunk $\overline{N_0} \cong N_t$ is not a solid torus. Then L_t is obtained as the union of the two irreducible manifolds M_t and N_t along their boundaries (2.14), none of them being a solid torus. Therefore L_t is irreducible (see [15], General Principle).

Assume first that there exists a connected component $M_t(\sigma)$ of M_t whose boundary is not connected. Let G_0 (resp. G_t) be the normalized plumbing graph of $\overline{L_0}$ (resp. of L_t). Gluing the manifold $M_t(\sigma)$ to the trunk N_t increases the number of cycles in the plumbing graph G_0 . Therefore, $\text{rank } H_1(G_0, \mathbb{Z}) < \text{rank } H_1(G_t, \mathbb{Z})$ and $\overline{L_0}$ is not homeomorphic to L_t .

We now assume that each connected component $M_t(\sigma)$ of M_t has a connected boundary, i.e. that $M_t(\sigma) \cap N_t$ consist of a single torus T_σ .

Let \mathcal{T}_t be the separating family of tori of the minimal Jaco-Shalen-Johannson (JSJ) decomposition of L_t (see [7], 1.9). Then \mathcal{T}_t decomposes as the union

$$\mathcal{T}_t = \mathcal{T}_0 \cup \mathcal{T} \cup \mathcal{T}',$$

where :

- \mathcal{T}_0 is the minimal separating family of the trunk $\overline{N_t}$,
- \mathcal{T} is the minimal separating family of M_t ,
- \mathcal{T}' is the union of tori in $M_t \cap N_t$ defined as follows : $T_\sigma \subset \mathcal{T}'$ if and only if the Seifert structure of $M_t(\sigma)$ on T_σ is not homological to that of N_t .

As $\overline{N_0} \cong N_t$ is the complement of a tubular neighborhood of the link $\overline{K_0}$ in $\overline{L_0}$, then \mathcal{T}_0 is also a separating family for $\overline{L_0}$. Let $\mathcal{T}_0^{\min} \subset \mathcal{T}_0$ be the minimal separating family for $\overline{L_0}$.

If $\mathcal{T} \cup \mathcal{T}' \neq \emptyset$, then the number of tori in \mathcal{T}_t is strictly greater than that in \mathcal{T}_0^{\min} . Therefore L_t is not homeomorphic to $\overline{L_0}$.

Let us assume $\mathcal{T} \cup \mathcal{T}' = \emptyset$. Then each $M_t(\sigma)$ is a Seifert manifold and the Seifert structure of N_t is homologically equivalent to that of $M_t(\sigma)$ on T_σ . We now use the following :

Remark. Let M be an irreducible orientable 3-dimensional manifold whose JSJ decomposition admits only Seifert pieces with orientable basis. Assume that M is not diffeomorphic neither to a lens space nor to a solid torus. Then, according to the classical classification of irreducible 3-dimensional manifolds (see [7]), the following two numbers are some numerical invariants of the homeomorphism class of M :

- (1) The sum $g(M)$ of the genus of the bases of the Seifert pieces of M in any JSJ decomposition of M ,
- (2) the global number $s(M)$ of exceptional Seifert leaves in the minimal decomposition of M .

Let r be the number of irreducible components of $\Sigma(f)$. As each connected component $M_t(\sigma)$ of M_t has a connected boundary, then r is also the number of irreducible components of the curve $\overline{\Sigma(f)}$. Therefore the trunk $N_0 \cong N_t$ has r boundary components (Corollary 2.8), and $\overline{L_0}$ is obtained by gluing r solid tori along the r boundary components of $\overline{N_0}$. We then have :

$$g(\overline{L_0}) = g(\overline{N_0}) \text{ and } s(\overline{L_0}) \leq s(\overline{N_0}) + r \quad (*)$$

Let σ be an irreducible component of $\Sigma(f)$. Let $p : M_t(\sigma) \rightarrow \mathbb{S}^1$ be the locally trivial fibration with fiber $F_{t,a}$ and monodromy $h : F_{t,a} \rightarrow F_{t,a}$ defined in 4.17.

If the transversal section of F_0 at a point of $\sigma \setminus \{0\}$ is the ordinary quadratic germ, then the Milnor fibre $F_{t,a}$ is an annulus $[-1, +1] \times \mathbb{S}^1$. As $M_t(\sigma)$ has a connected boundary, then $h : [-1, +1] \times \mathbb{S}^1 \rightarrow [-1, +1] \times \mathbb{S}^1$ is isotopic to the diffeomorphism $h(t, z) = (-t, \bar{z})$ and its mapping torus $M_t(\sigma)$ is the so-called Seifert Q manifold ([22]), which has two exceptional fibers and base a disk.

In all other cases, $\chi(F_{t,a}) < 0$. Then $M_t(\sigma)$ has either $g(M_t(\sigma)) > 0$ or at least two exceptional fibers, i.e. $s(M_t(\sigma)) \geq 2$.

If there exists σ such that $g(M_t(\sigma)) > 0$, then $g(L_t) > g(N_t) = g(\overline{N_0}) = g(\overline{L_0})$, then L_t is not homeomorphic to $\overline{L_0}$.

Otherwise, each $M_t(\sigma)$ has at least 2 exceptional fibres, and

$$s(L_t) \geq s(N_t) + 2r$$

Then (*) implies $s(L_t) > s(\overline{L_0})$ and L_t is not homeomorphic to $\overline{L_0}$.

□

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